Finding Equations of Polynomial Functions with Given Zeros

Polynomials are functions of general form → $P(x) = a_n x^n + a_{n-1}\, x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$

$(n \in \textit{whole \#'s})$

Polynomials can also be written in factored form → $P(x) = a(x - z_1)(x - z_2) \ldots (x - z_i) \quad (a \in \mathbb{R})$

Given a list of "zeros", it is possible to find a polynomial function that has these specific zeros. In fact, there are multiple polynomials that will work. In order to determine an ***exact polynomial***, the "zeros" ***and a point*** on the polynomial must be provided.

Examples: Practice finding polynomial equations in general form with the given zeros.

Find an* equation of a polynomial with the following two zeros: $x = -2, x = 4$

Denote the given zeros as z_1 *and* z_2

Step 1: Start with the factored form of a polynomial.

$$P(x) = a(x - z_1)(x - z_2)$$

Step 2: Insert the given zeros and simplify.

$$P(x) = a(x - (-2))(x - 4)$$

$$P(x) = a(x + 2)(x - 4)$$

Step 3: Multiply the factored terms together.

$$P(x) = a(x^2 - 2x - 8)$$

Step 4: The answer can be left with the generic "a", or a value for "a" can be chosen, inserted, and distributed.

i.e. if $a = 1,\ then\ P(x) = x^2 - 2x - 8$

i.e. if $a = -2,\ then\ P(x) = -2x^2 + 4x + 16$

**Each different choice for "a" will result in a distinct polynomial. Thus, there are an infinite number of polynomials with the two zeros $x = -2$ and $x = 4$.*

Find the equation of a polynomial with the following zeroes: $x = 0, -\sqrt{2}, \sqrt{2}$ that goes through the point $(-2, 1)$.

Denote the given zeros as z_1, z_2 *and* z_3

Step 1: Start with the factored form of a polynomial.

$$P(x) = a(x - z_1)(x - z_2)(x - z_3)$$

Step 2: Insert the given zeros and simplify.

$$P(x) = a(x - 0)(x - (-\sqrt{2}))(x - \sqrt{2})$$

$$P(x) = ax(x + \sqrt{2})(x - \sqrt{2})$$

Step 3: Multiply the factored terms together

$$P(x) = a(x^3 - 2x)$$

Step 4: Insert the given point $(-2, 1)$ to solve for "a".

$$1 = a[(-2)^3 - 2(-2)]$$
$$1 = a[-8 + 4]$$
$$1 = -4a$$
$$a = -\tfrac{1}{4}$$

Step 5: Insert the value for *"a" into the polynomial,* distribute, and re-write the function.

$$P(x) = -\tfrac{1}{4}(x^3 - 2x) = -\tfrac{1}{4}x^3 + \tfrac{1}{2}x$$

Polynomials can have zeros with *multiplicities greater than 1*. This is easier to see if the Polynomial is written in factored form.

Insert Multiplicities of each zero

$$P(x) = a(x - z_1)^m(x - z_2)^n \dots (x - z_i)^p$$

Insert zeros

Multiplicity - The number of times a "zero" is repeated in a polynomial. The multiplicity of each zero is inserted as an exponent of the factor associated with the zero. If the multiplicity is not given for a zero, it is assumed to be 1.

Examples: Practice finding polynomial equations with the given zeros and multiplicities.

Find an equation of a polynomial with the given zeroes and associated multiplicities. *Leave the answer in factored form.*

Zeros	Multiplicity
$x = 1$	2
$x = -2$	3
$x = 3$	1

Step 1: Write the factored form of the Polynomial.

$$P(x) = a(x - z_1)^m(x - z_2)^n(x - z_3)^p$$

Step 2: Insert the given zeros and their corresponding multiplicities.

$$P(x) = a(x - 1)^2(x - -2)^3(x - 3)^1$$

Step 3: Simplify any algebra if necessary. The answer can be left with the generic "a", or a specific value for "a" can be chosen and inserted if requested.

$$P(x) = a(x - 1)^2(x + 2)^3(x - 3)$$

i.e. $let\ a = 1,\ then$

$$P(x) = (x - 1)^2(x + 2)^3(x - 3)$$

$let\ a = -2,\ then$

$$P(x) = -2(x - 1)^2(x + 2)^3(x - 3)$$

Find an equation of a polynomial with the given zeros and associated multiplicities. *Expand the answer into general form.*

Zeros can be real or imaginary

Zeros	Multiplicity
$x = 0$	3
$x = -1$	2
$x = i$	1
$x = -i$	1

Step 1: Write the factored form of the Polynomial.

$$P(x) = a(x - z_1)^m(x - z_2)^n \dots (x - z_i)^p$$

Step 2: Insert the given zeros and their corresponding multiplicities.

$$P(x) = a(x - 0)^3(x - -1)^2(x - i)^1(x - -i)^1$$

Step 3: Simplify any algebra if necessary.

$$P(x) = ax^3(x + 1)^2(x - i)(x + i)$$

Step 4: Multiply the factored terms together. Recall that $i^2 = -1$! Note the generic "a" can be used and distributed, or a specific value for "a" can be chosen and inserted if requested.

$$P(x) = ax^7 + 2ax^6 + 2ax^5 + 2ax^4 + ax^3$$

Practice Problems: **Try these problems on your own!**

Find **an** equation of a Polynomial with the given zeros.

1. Zeros: $x = -1, -3$ *Answer:* $f(x) = a(x^2 + 4x + 3)$

2. Zeros: $x = -2,\ 2, \sqrt{3}, -\sqrt{3}$ *Answer:* $f(x) = a(x^4 - 7x^2 + 12)$

3. Zeros: $x = -4i, 4i$ *Answer:* $f(x) = a(x^2 + 16)$

Find **the** equation of a Polynomial given the following zeros and a point on the Polynomial.

4. Zeros: $x = 0, -4$ Point: $(-3\ , 6)$ *Answer:* $f(x) = -2x^2 - 8x$

5. Zeros: $x = -2,\ 5$ Point: $(2\ , -3)$ *Answer:* $f(x) = \frac{1}{4}x^2 - \frac{3}{4}x - \frac{5}{2}$

6. Zeros: $x = -4, -1,\ 1$ Point: $(2\ , 9)$ *Answer:* $f(x) = \frac{1}{2}x^3 + 2x^2 - \frac{1}{2}x - 2$

Find **an** equation of a Polynomial given the following zeros with the listed multiplicities. In each example, set $a = 1$.

7. Zero: $x = 3,\ Multiplicity\ 2$ *Answer:* $f(x) = x^2 - 6x + 9$

8. Zero: $x = 0,\ Multiplicity\ 2$ *Answer:* $f(x) = x^3 + 2x^2$
   Zero: $x = -2,\ Multiplicy\ 1$

9. Zero: $x = -1,\ Multiplicity\ 1$ *Answer:* $f(x) = x^4 - 1$
   Zero: $x = 1,\ Multiplicy\ 1$
   Zero: $x = -i,\ Multiplicy\ 1$
   Zero: $x = -i,\ Multiplicy\ 1$

10. Zero: $x = -3,\ Multiplicity\ 2$ *Answer:* $f(x) = x^3 + 4x^2 - 3x - 18$
    Zero: $x = 2,\ Multiplicy\ 1$