Unit 8.1-8.2 Long Division of Polynomials NOTES

Dividing Polynomials; Remainder and Factor Theorems

In this section we will learn how to divide polynomials, an important tool needed in factoring them. This will begin our *algebraic* study of polynomials.

Dividing by a Monomial:

Recall from the previous section that a monomial is a single term, such as $6x^3$ or -7. To divide a polynomial by a monomial, divide each term in the polynomial by the monomial, and then write each quotient in lowest terms.

Example 1: Divide $9x^4 + 3x^2 - 5x + 6$ by $3x$.

Solution:

Step 1: Divide each term in the polynomial $9x^4 + 3x^2 - 5x + 6$ by the monomial $3x$.

$$\frac{9x^4+3x^2-5x+6}{3x} = \frac{9x^4}{3x} + \frac{3x^2}{3x} - \frac{5x}{3x} + \frac{6}{3x}$$

Step 2: Write the result in lowest terms.

$$\frac{9x^4}{3x} + \frac{3x^2}{3x} - \frac{5x}{3x} + \frac{6}{3x} = 3x^3 + x - \frac{5}{3} + \frac{2}{x}$$

Thus, $9x^4 + 3x^2 - 5x + 6$ divided by $3x$ is equal to $3x^3 + x - \frac{5}{3} + \frac{2}{x}$

Long Division of Polynomials:

To divide a polynomial by a polynomial that is not a monomial we must use long division. Long division for polynomials is very much like long division for numbers. For example, to divide $3x^2 - 17x - 25$ (the **dividend**) by $x - 7$ (the **divisor**), we arrange our work as follows.

quotient ↓

$$
\begin{array}{rll}
 & 3x+4 & \\
\text{divisor} \rightarrow \quad x-7 & \overline{)\,3x^2-17x-25} & \leftarrow \text{dividend} \\
 & \underline{3x^2-21x} & \text{Multiply divisor by } 3x \\
 & \quad 4x-25 & \text{Subtract, then "bring down" } -25 \\
 & \quad \underline{4x-28} & \text{Multiply divisor by } 4 \\
 & \qquad 3 \quad \leftarrow \text{remainder} & \text{Subtract}
\end{array}
$$

The division process ends when the last line is of lesser degree than the divisor. The last line then contains the **remainder**, and the top line contains the **quotient**. The result of the division can be interpreted in either of two ways

$$\frac{3x^2-17x-25}{x-7}=3x+4+\frac{3}{x-7}$$

or

$$3x^2-17x-25=(x-7)(3x+4)+3$$

We summarize what happens in any long division problem in the following theorem.

Division Algorithm:

If $P(x)$ and $D(x)$ are polynomials, with $D(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = D(x) \cdot Q(x) + R(x)$$

where $R(x)$ is either 0 or of less degree than the degree of $D(x)$. The polynomials $P(x)$ and $D(x)$ are called the **dividend** and the **divisor**, respectively, $Q(x)$ is the **quotient**, and $R(x)$ is the **remainder**.

Example 2: Let $P(x) = 3x^2 + 17x + 10$ and $D(x) = 3x + 2$. Using long division, find polynomials $Q(x)$ and $R(x)$ such that $P(x) = D(x) \cdot Q(x) + R(x)$.

Solution:

Step 1: Write the problem, making sure that both polynomials are written in descending powers of the variables.

$$3x+2\,\overline{)\,3x^2+17x+10}$$

Example 2 (Continued):

Step 2: Divide the first term of $P(x)$ by the first term of $D(x)$.

Since $\dfrac{3x^2}{3x} = x$, place this result above the division line.

$$\begin{array}{r l}
x & \leftarrow \text{Result of } \dfrac{3x^2}{3x} \\
3x+2\overline{)3x^2+17x+10} &
\end{array}$$

Step 3: Multiply $3x + 2$ and x, and write the result below $3x^2 + 17x + 10$.

$$\begin{array}{r l}
x & \\
3x+2\overline{)3x^2+17x+10} & \\
\underline{3x^2+\ \ 2x}\qquad\quad & \leftarrow x(3x+2) = 3x^2 + 2x
\end{array}$$

Step 4: Now subtract $3x^2 + 2x$ from $3x^2 + 17x$. Do this by mentally changing the signs on $3x^2 + 2x$ and adding.

$$\begin{array}{r l}
x & \\
3x+2\overline{)3x^2+17x+10} & \\
\underline{3x^2+\ \ 2x}\qquad\quad & \\
15x\qquad\quad & \leftarrow \text{Subtract}
\end{array}$$

Step 5: Bring down 10 and continue by dividing $15x$ by $3x$.

$$\begin{array}{r l}
x+5 & \leftarrow \dfrac{15x}{3x} = 5 \\
3x+2\overline{)3x^2+17x+10} & \\
\underline{3x^2+\ \ 2x}\qquad\quad & \\
15x+10 & \leftarrow \text{Bring down 10} \\
\underline{15x+10} & \leftarrow 5(3x+2) = 15x+10 \\
0 & \leftarrow \text{Subtract}
\end{array}$$

Example 2 (Continued):

Step 6: The process is complete at this point because we have a zero in the final row. From the long division table we see that $Q(x) = x + 5$ and $R(x) = 0$, so

$$3x^2 + 17x + 10 = (3x + 2)(x + 5) + 0$$

Note that since there is no remainder, this quotient could have been found by factoring and writing in lowest terms.

Example 3: Find the quotient and remainder of $\dfrac{4x^3 - 3x - 2}{x+1}$ using long division.

Solution:

Step 1: Write the problem, making sure that both polynomials are written in descending powers of the variables. Add a term with 0 coefficient as a place holder for the missing x^2 term.

$$x+1\overline{)4x^3 + 0x^2 - 3x - 2} \quad \leftarrow \text{Missing term } (0x^2)$$

Step 2: Start with $\dfrac{4x^3}{x} = 4x^2$.

$$\begin{array}{rll} & 4x^2 & \leftarrow \dfrac{4x^3}{x} = 4x^2 \\ x+1 & \overline{)4x^3 + 0x^2 - 3x - 2} & \\ & \underline{4x^3 + 4x^2} & \leftarrow 4x^2(x+1) \end{array}$$

Step 3: Subtract by changing the signs on $4x^3 + 4x^2$ and adding. Then Bring down the next term.

$$\begin{array}{rll} & 4x^2 & \\ x+1 & \overline{)4x^3 + 0x^2 - 3x - 2} & \\ & \underline{4x^3 + 4x^2} & \\ & \quad -4x^2 - 3x & \leftarrow \text{Subtract and bring down } -3x \end{array}$$

Example 3 (Continued):

Step 4: Now continue with $\frac{-4x^2}{x} = -4x$.

$$
\begin{array}{r|l}
 & \;\;4x^2 - 4x \quad \leftarrow \frac{-4x^2}{x} = -4x \\
x+1 & 4x^3 + 0x^2 - 3x - 2 \\
 & \underline{4x^3 + 4x^2} \\
 & \quad -4x^2 - 3x \\
 & \quad \underline{-4x^2 - 4x} \quad \leftarrow -4x(x+1) \\
 & \qquad\qquad x - 2 \leftarrow \text{Subtract and bring down } -2
\end{array}
$$

Step 5: Finally, $\frac{x}{x} = 1$.

$$
\begin{array}{r|l}
 & \;\;4x^2 - 4x + 1 \quad \leftarrow \frac{x}{x} = 1 \\
x+1 & 4x^3 + 0x^2 - 3x - 2 \\
 & \underline{4x^3 + 4x^2} \\
 & \quad -4x^2 - 3x \\
 & \quad \underline{-4x^2 - 4x} \\
 & \qquad\qquad x - 2 \\
 & \qquad\qquad \underline{x + 1} \leftarrow 1(x+1) \\
 & \qquad\qquad\; -3 \leftarrow \text{Subtract}
\end{array}
$$

Step 6: The process is complete at this point because -3 is of lesser degree than the divisor $x + 1$. Thus, the quotient is $4x^2 - 4x + 1$ and the remainder is -3, and

$$\frac{4x^3 - 3x - 2}{x+1} = 4x^2 - 4x + 1 + \frac{-3}{x+1}.$$