# 7.5 Operations with Matrices

# What You Should Learn

- Decide whether two matrices are equal.
- Add and subtract matrices and multiply matrices by scalars.
- Multiply two matrices.
- Use matrix operations to model and solve real-life problems.

# Equality of Matrices

# Equality of Matrices

This section introduces some fundamentals of matrix theory. It is standard mathematical convention to represent matrices in any of the following three ways.

**Representation of Matrices**

1. A matrix can be denoted by an uppercase letter such as

   $A$, $B$, or $C$.

2. A matrix can be denoted by a representative element enclosed in brackets, such as

   $[a_{ij}]$, $[b_{ij}]$, or $[c_{ij}]$.

3. A matrix can be denoted by a rectangular array of numbers such as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

# Equality of Matrices

Two matrices

$$A = [a_{ij}] \quad \text{and} \quad B = [b_{ij}]$$

are **equal** when they have the same dimension ($m \times n$) and all of their corresponding entries are equal.

# Example 1 – *Equality of Matrices*

Solve for $a_{11}$, $a_{12}$, $a_{21}$, and $a_{22}$ in the following matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$$

Solution:

Because two matrices are equal only when their corresponding entries are equal, you can conclude that

$a_{11} = 2$, $a_{12} = -1$, $a_{21} = -3$, and $a_{22} = 0$.

# Equality of Matrices

Be sure you see that for two matrices to be equal, they must have the same dimension *and* their corresponding entries must be equal.

For instance,

$$\begin{bmatrix} 2 & -1 \\ \sqrt{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & 0.5 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}.$$

# Matrix Addition and Scalar Multiplication

# Matrix Addition and Scalar Multiplication

You can add two matrices (of the same dimension) by adding their corresponding entries.

**Definition of Matrix Addition**

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of dimension $m \times n$, then their sum is the $m \times n$ matrix given by $A + B = [a_{ij} + b_{ij}]$. The sum of two matrices of different dimensions is undefined.

# Example 2 – *Addition of Matrices*

**a.** $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 + 1 & 2 + 3 \\ 0 + (-1) & 1 + 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$

**b.** $\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

**c.** The sum of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$

is undefined because $A$ is of dimension $2 \times 3$ and $B$ is of dimension $2 \times 2$.

# Matrix Addition and Scalar Multiplication

In operations with matrices, numbers are usually referred to as **scalars.** In this text, scalars will always be real numbers. You can multiply a matrix *A* by a scalar *c* by multiplying each entry in *A* by *c*.

**Definition of Scalar Multiplication**

If $A = [a_{ij}]$ is an $m \times n$ matrix and $c$ is a scalar, then the **scalar multiple** of $A$ by $c$ is the $m \times n$ matrix given by $cA = [ca_{ij}]$.

# Example 3 – *Scalar Multiplication*

Find 3*A* *using* $A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$.

Solution:

$$3A = 3\begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(2) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

# Matrix Addition and Scalar Multiplication

The properties of matrix addition and scalar multiplication are similar to those of addition and multiplication of real numbers.

One important property of addition of real numbers is that the number 0 is the additive identity.

That is, $c + 0 = c$ for any real number $c$. For matrices, a similar property holds.

# Matrix Addition and Scalar Multiplication

That is, if $A$ is an $m \times n$ matrix and $O$ is the $m \times n$ **zero matrix** consisting entirely of zeros, then $A + O = A$.

In other words, $O$ is the **additive identity** for the set of all $m \times n$ matrices. For example, the following matrices are the additive identities for the sets of all $2 \times 3$ and $2 \times 2$ matrices.

$$O = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{2 \times 3 \text{ zero matrix}} \quad \text{and} \quad O = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{2 \times 2 \text{ zero matrix}}$$

# Matrix Addition and Scalar Multiplication

## Properties of Matrix Addition and Scalar Multiplication

Let $A$, $B$, and $C$ be $m \times n$ matrices and let $c$ and $d$ be scalars.

| | |
|---|---|
| **1.** $A + B = B + A$ | Commutative Property of Matrix Addition |
| **2.** $A + (B + C) = (A + B) + C$ | Associative Property of Matrix Addition |
| **3.** $(cd)A = c(dA)$ | Associative Property of Scalar Multiplication |
| **4.** $1A = A$ | Scalar Identity |
| **5.** $A + O = A$ | Additive Identity |
| **6.** $c(A + B) = cA + cB$ | Distributive Property |
| **7.** $(c + d)A = cA + dA$ | Distributive Property |

# Example 5 – *Using the Distributive Property*

$$3\left(\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}\right) = 3\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + 3\begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 0 \\ 12 & 3 \end{bmatrix} + \begin{bmatrix} 12 & -6 \\ 9 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -6 \\ 21 & 24 \end{bmatrix}$$

# Matrix Addition and Scalar Multiplication

The algebra of real numbers and the algebra of matrices have many similarities. For example, compare the following solutions.

| *Real Numbers* | *m × n Matrices* |
|---|---|
| *(Solve for x.)* | *(Solve for X.)* |
| $x + a = b$ | $X + A = B$ |
| $x + a + (-a) = b + (-a)$ | $X + A + (-A) = B + (-A)$ |
| $x + 0 = b - a$ | $X + O = B - A$ |
| $x = b - a$ | $X = B - A$ |

# Example 6 – *Solving a Matrix Equation*

Solve for $X$ in the equation

$$3X + A = B,$$

where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

# Example 6 – *Solution*

Begin by solving the matrix equation for *X*.

$$3X + A = B \quad \Rightarrow \quad 3X = B - A \quad \Rightarrow \quad X = \tfrac{1}{3}(B - A)$$

Now, using the matrices *A* and *B*, you have

$$X = \frac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right)$$

Substitute the matrices.

$$= \frac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix}$$

Subtract matrix *A* from matrix *B*.

$$= \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}.$$

Multiply the resulting matrix by $\frac{1}{3}$.

# Matrix Multiplication

# Matrix Multiplication

Another basic matrix operation is **matrix multiplication**. You will see later, however, that the definition of the product of two matrices has many practical applications.

**Definition of Matrix Multiplication**

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the product $AB$ is an $m \times p$ matrix given by

$$AB = [c_{ij}]$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdot \cdot \cdot + a_{in}b_{nj}.$$

# Matrix Multiplication

The definition of matrix multiplication indicates a *row-by-column* multiplication, where the entry in the $i$th row and $j$th column of the product $AB$ is obtained by multiplying the entries in the $i$th row of $A$ by the corresponding entries in the $j$th column of $B$ and then adding the results.

# Matrix Multiplication

The general pattern for matrix multiplication is as follows.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3j} & \cdots & b_{3p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

$$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = c_{ij}$$

# Example 7 – *Finding the Product of Two Matrices*

Find the product *AB* using $A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$.

Solution:

First, note that the product *AB* is defined because the number of columns of *A* is equal to the number of rows of *B*. Moreover, the product *AB* has dimension $3 \times 2$.

To find the entries of the product, multiply each row of *A* by each column of *B.*

# Example 7 – *Solution*

cont'd

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(-3) + (3)(-4) & (-1)(2) + (3)(1) \\ (4)(-3) + (-2)(-4) & (4)(2) + (-2)(1) \\ (5)(-3) + (0)(-4) & (5)(2) + (0)(1) \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

# Matrix Multiplication

## Properties of Matrix Multiplication

Let $A$, $B$, and $C$ be matrices and let $c$ be a scalar.

1. $A(BC) = (AB)C$ — Associative Property of Matrix Multiplication
2. $A(B + C) = AB + AC$ — Left Distributive Property
3. $(A + B)C = AC + BC$ — Right Distributive Property
4. $c(AB) = (cA)B = A(cB)$ — Associative Property of Scalar Multiplication

# Matrix Multiplication

### Definition of Identity Matrix

The $n \times n$ matrix that consists of 1's on its main diagonal and 0's elsewhere is called the **identity matrix of dimension $n \times n$** and is denoted by

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

Identity matrix

Note that an identity matrix must be *square*. When the dimension is understood to be $n \times n$, you can denote $I_n$ simply by $I$.

# Matrix Multiplication

If $A$ is an $n \times n$ matrix, then the identity matrix has the property that $AI_n = A$ and $I_nA = A$.

For example,

$$\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} \qquad AI = A$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix}. \qquad IA = A$$

# Applications

# Applications

Matrix multiplication can be used to represent a system of linear equations. Note how the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

can be written as the matrix equation

$$AX = B$$

# Applications

where *A* is the *coefficient matrix* of the system, and *X* and *B* are column matrices. The column matrix *B* is also called a *constant matrix*. Its entries are the constant terms in the system of equations.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A \times X = B$$

# Example 10 – *Solving a System of Linear Equations*

For the system of linear equations, (a) write the system as a matrix equation $AX = B$ and (b) use Gauss-Jordan elimination on $[A \vdots B]$ to solve for the matrix $X$.

$$\begin{cases} x_1 - 2x_2 + x_3 = -4 \\ x_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 - 2x_3 = 2 \end{cases}$$

# Example 10 – *Solution*

**a.** In matrix form *AX* = *B,* the system can be written as follows.

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$$

**b.** The augmented matrix is

$$[A \,\vdots\, B] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 0 & 1 & 2 & 4 \\ 2 & 3 & -2 & 2 \end{array}\right].$$

# Example 10 – *Solution*

cont'd

Using Gauss-Jordan elimination, you can rewrite this equation as

$$[I \vdots X] = \begin{bmatrix} 1 & 0 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}.$$

So, the solution of the system of linear equations is

$x_1 = -1$, $x_2 = 2$, and $x_3 = 1$.

The solution of the matrix equation is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

# Homework:

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